Final Exam Solutions

1. Let f be a differentiable function with $f(1,3)=7$ and $\nabla f(1,3)=\langle 5,-2\rangle$.

   (a) Find the directional derivative of f at $(1,3)$ in the direction of the point $(4,2)$. (5 pts)

   The unit vector in the direction of $(4,2)$ is $\langle \frac{3}{\sqrt{10}}, -\frac{1}{\sqrt{10}}\rangle$. Dot with the gradient to get the directional derivative is $\langle \frac{3}{\sqrt{10}}, -\frac{1}{\sqrt{10}}\rangle \cdot \langle 5,-2\rangle = \frac{17}{\sqrt{10}}$.

   (b) Find a reasonable approximation for $f(.9,3.1)$. (5 pts)

   $f(.9,3.1) \approx f(1,3)+f_x(1,3)(.9-1)+f_y(1,3)(3.1-3) =$
   $7+(5)(-.1)+(-2)(.1)=6.3$

   (c) Let $z=f(\sin(st)+1, e^s+t)$. Find $\partial z/\partial s$ when $s=0, t=2$. (10 pts)

   Let $x=\sin(st)+1, y=e^s+t$. By the chain rule,
   $\partial z/\partial s = f_x(x,y)(t\cos(st))+f_y(x,y)(e^s)$. At $s=0,t=2$ we have that $x=1,y=3$ so $\partial z/\partial s = f_x(1,3)(2)+f_y(1,3)(1)=5(2)+(-2)(1)=8$.

2. Evaluate the limit or show that it does not exist. (10 pts)

$$\lim_{(x,y)\to(0,0)} \frac{x^3y^2-xy^4}{x^5+y^5}$$

   Along the paths $x=0, y=0, y=x$ the limit is 0, however along $y=2x$ the limit is $-\frac{12}{33}$ so the limit does not exist.

3. Find the absolute maximum and minimum of $f(x,y,z)=x+y^2+3z$ on the paraboloid $x^2+2y^2+3z^2=36$. (15 pts)

   This is a closed and bounded region so there is a max and min. There is no interior so just check for critical points on the paraboloid using Lagrange multipliers. The Lagrange multiplier equations are
   $1=\lambda 2x, 2y=\lambda 4y, 3=\lambda 6z, x^2+2y^2+3z^2=36$. The first and third equations simplify to $x=1/(2\lambda), z=1/(2\lambda)$ so $x=z$. The second equation gives us that $y=0$ or $\lambda=1/2$. Consider each case and use the fourth equation to get that there are 4 critical points: $(3,0,3)$, $(-3,0,-3)$, $(1,4,1)$, and $(1,-4,1)$. Plugging these into f we get $f(3,0,3)=12$, $f(-3,0,-3)=-12$, $f(1,4,1)=20$, $f(1,-4,1)=20$ so the maximum is 20 and the minimum is -12.

4. Compute $\int_C (y^2e^{xy})dx+(e^{xy}+xye^{xy})dy$ where C is the curve consisting of the two line segments from $(0,0)$ to $(2,2)$ and from $(2,2)$ to $(0,5)$. (10 pts)

   This can be done two different ways. If $F=\langle P,Q\rangle = \langle y^2e^{xy}, e^{xy}+xye^{xy}\rangle$ then $P_y=2ye^{xy}+xy^2e^{xy}, Q_x=2ye^{xy}+xy^2e^{xy}$ so F is conservative. It has potential function $f(x,y) = ye^{xy}$ so by the fundamental theorem of line integrals, the integral is $f(0,5) - f(0,0) = 5$.

The other way to do this is to close the region with the line segment from $(0,0)$ to $(0,5)$ and use Green's theorem. The integral over the whole triangle is 0 by Green's Theorem as $Q_x - P_y = 0$. The curve from $(0,0)$ to $(0,5)$ can be parametrized as $x = 0, y = t, 0 \leq t \leq 5, dx = 0, dy = dt$ so the integral over this line segment is $\int_0^5 1\ dt = 5$. Combine these two facts to get that the integral is 5.

5. Let D be a region on the xy-plane. Let S be the part of the plane $4x - 6y + 2z = 5$ which lies above or below the region D, (i.e. points on the plane $4x - 6y + 2z = 5$ with (x,y) in D). If the area of S is 11, find the area of D. (10 pts)

   The surface S can be parametrized as $r(x,y) = \langle x, y, (5/2) - 2x + 3y\rangle$ where the possible (x,y) values are exactly those in D. Then $r_x \times r_y = \langle 2, -3, 1\rangle$ so $|r_x \times r_y| = \sqrt{14}$. Using the surface area formula we get that the surface area of S is $A(S) = \iint_D \sqrt{14}\ dA = \sqrt{14}A(D)$ where $A(D)$ is the area of D. Set $A(S) = 11$ and solve for $A(D)$ to get $A(D) = 11/\sqrt{14}$.

6. Let S be the boundary of the region which is both inside the sphere $x^2 + y^2 + z^2 = 8$ and above the cone $z = \sqrt{x^2 + y^2}$. Find $\iint_S F \cdot\ d\mathbf{S}$ where $F(x,y,z) = \langle xz + 5y^2, e^{\cos(xz)}, z^2\rangle$. (15 pts)

   Use the divergence theorem. The divergence of F is $3z$. The region can be set up in either cylindrical or spherical coordinates and two set-ups are the following:

$$\int_0^{2\pi} \int_0^2 \int_r^{\sqrt{8-r^2}} 3zr\ dz dr d\theta$$

$$\int_0^{2\pi} \int_0^{\pi/4} \int_0^{\sqrt{8}} 3\rho^3 \sin(\phi)\cos(\phi)\ d\rho d\phi d\theta\ .$$

   The value of the integral is 24π.

7. Find $\int_C F \cdot dr$ where C is the intersection of the plane $z = 1 - 2x - 3y$ and the cylinder $x^2 + y^2 = 4$ oriented clockwise when viewed from above and $F(x,y,z) = \langle yz + \cos(x^2), -x^2, 3y\rangle$. (20 pts)

   Use Stokes Theorem with S the part of the plane which is inside the cylinder, oriented down. S can be parametrized as $r(x,y) = \langle x, y, 1 - 2x - 3y$ where $x^2 + y^2 \leq 4$. Then $r_x \times r_y = \langle 2, 3, 1\rangle$ and we change this to $\langle -2, -3, -1\rangle$ to match the orientation. The curl of F is $\langle 3, y, -2x - z\rangle$ and on S this is $\langle 3, y, 3y - 1\rangle$. The dot product of the curl and the normal vector is $-5 - 6y$ so the integral is $\iint_{x^2+y^2\leq 4} -5 - 6y\ dA$. To evaluate, switch to polar to get $\int_0^{2\pi} \int_0^2 -5r - 6r^2 \sin(\theta)\ dr d\theta = -20\pi$.